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Prehomogeneous spaces for the coadjoint action of a parabolic group

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Abstract

Let k be an algebraically closed field and let G be a reductive linear algebraic group over k . Let P be a parabolic subgroup of G , P_u its unipotent radical and \mathfrak{p}_u the Lie algebra of P_u . A fundamental result of R. Richardson says that P acts on \mathfrak{p}_u with a dense orbit (see [R.W. Richardson, Bull. London Math. Soc. 6 (1974) 21–24]). The analogous result for the coadjoint action of P on \mathfrak{p}_u^* is already known for $\text{char } k = 0$ (see [A. Joseph, J. Algebra 48 (1977) 241–289]). In this note we prove this result for arbitrary characteristic. Our principal result is that \mathfrak{b}_u^* is a prehomogeneous space for a Borel subgroup B of G . From this we deduce that a parabolic subgroup P of G acts on \mathfrak{n}^* with a dense orbit for any P -submodule \mathfrak{n} of P . Further, we determine when the orbit map for such an orbit is separable.

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1. Introduction

Let G be a reductive linear algebraic group defined over the algebraically closed field k . Let P be a parabolic subgroup of G , P_u its unipotent radical and $\mathfrak{p}_u = \text{Lie}(P_u)$ the Lie algebra of P_u . We consider the coadjoint action of P on \mathfrak{p}_u^* . Our principal result is

Theorem 1.1. *Let G be a reductive group and let B be a Borel subgroup of G . Then \mathfrak{b}_u^* is a prehomogeneous space for B .*

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From which we deduce

Corollary 1.2. *Let G be a reductive group, let P be a parabolic subgroup of G and \mathfrak{n} a P -submodule of \mathfrak{p}_u . Then \mathfrak{n}^* is a prehomogeneous space for P .*

In [9], Richardson proved that P always acts on \mathfrak{p}_u with a dense orbit. Corollary 1.2 implies the analogous result for the coadjoint action of P on \mathfrak{p}_u^* .

Theorem 1.1 is already known for $\text{char } k = 0$ (see [5, 2.4]). The proof in *loc. cit.* deals primarily with Lie algebras. The set $M = \{(f, x) \in \mathfrak{b}_u^* \times \mathfrak{b}_u : f([b, x]) = 0\}$ is considered and it is shown that there exists $f \in \mathfrak{b}_u^*$ such that $\{x \in \mathfrak{b}_u : (f, x) \in M\} = \emptyset$. For such f , $B \cdot f$ is shown to be dense in \mathfrak{b}_u^* .

If $\text{char } k \neq 0$, then for $f \in \mathfrak{b}_u^*$ the orbit map $B \rightarrow B \cdot f$ need not be separable. This implies that the proof in [5] is not valid for positive characteristic. Our proof uses a similar inductive method for constructing representatives of a dense B -orbit on \mathfrak{b}_u^* and in fact we get the same representatives. However, our proof is group theoretic so is valid in any characteristic.

In [1], the coadjoint action of B on \mathfrak{b}^* is considered for $k = \mathbb{C}$. It is shown that B acts on \mathfrak{b}^* with a dense orbit if G is not of type A_r , D_r (r odd) or E_6 . Also in *loc. cit.* the maximal dimension of a B -orbit on \mathfrak{b}^* is calculated, in the case B does not act on \mathfrak{b}^* with a dense orbit. These results are also given in [12, §4]. With more work, our methods could give these results for arbitrary characteristic. However, we choose not to pursue this here.

For a Lie algebra \mathfrak{q} over an algebraically closed field of characteristic zero there has been interest in the index of \mathfrak{q} . This is defined by $\text{ind } \mathfrak{q} = \min_{x \in \mathfrak{q}^*} \dim \mathfrak{q}_x$ where \mathfrak{q}_x denotes the centraliser of x and the definition goes back to J. Dixmier [4, 11.1.6]. For an algebraic Lie algebra \mathfrak{q} , $\text{ind } \mathfrak{q}$ is zero precisely when Q acts on \mathfrak{q}^* with a dense orbit where Q is a connected algebraic group such that $\text{Lie}(Q) = \mathfrak{q}$. We refer the reader to the recent papers [6,7] of D. Panyushev for more details.

Let R be an algebraic group and V a rational R -module. A result of V. Pyasetskii [8, Corollary 2] says that if R acts on V with finitely many orbits, then R also acts on V^* with finitely many orbits and there is a natural bijection between the set of R -orbits on V and the set of R -orbits on V^* . This was generalised by G. Röhrle in [11, Theorem 1.4] which says that the modality of the action of R on V is equal to the modality of the action of R on V^* . These results suggest that there may be a natural bijection between the sets of R -orbits on V and V^* in general. From considering the map in [8] one suspects that such a bijection would reverse the closure order. However, there are many instances when a parabolic group P does not act on a P -submodule \mathfrak{n} with a dense orbit, for example, see [10]. Therefore, a bijection between the sets of R -orbits on V and V^* cannot reverse the closure order in general.

In Section 2 we give the notation that we require. Then in Section 3 we prove some lemmas we need for the proof of Theorem 1.1. The proofs of our main results are contained in Section 4, a lemma is required to prove Theorem 1.1 from which Corollary 1.2 is deduced. Table 1 in Section 4 gives a representative of the dense B -orbit on \mathfrak{b}_u^* for each simple G .

2. Notation

Let k be an algebraically closed field, R a linear algebraic group over k and V a rational R -module. The dual space V^* of V has the structure of an R -module via the contragredient action. For an R -submodule U of V , we write $\text{Ann}_{V^*}(U) = \{\phi \in V^* : \phi(u) = 0 \ \forall u \in U\}$. For $r \in R$ and $x \in V$, we write $r \cdot x$ for the image of x under r , $R \cdot x$ for the R -orbit of x and R_x for the stabiliser of x in R . We note that V is also a module for the Lie algebra $\mathfrak{r} = \text{Lie}(R)$ of R . For $s \in \mathfrak{r}$ and $x \in V$ we write $s \cdot x$ for the image of x under s and \mathfrak{r}_x for the stabiliser of x in \mathfrak{r} .

We recall that V is called a *prehomogeneous space* for R provided R acts on V with a dense orbit.

Let G be a reductive linear algebraic group over k . The rank of G is denoted by $\text{rank } G$. We write \mathfrak{g} or $\text{Lie}(G)$ for the Lie algebra of G ; likewise for closed subgroups of G . Fix a maximal torus T of G and let Ψ be the root system of G with respect to T . When G is simple we write ρ for the highest root of Ψ and when considering the (extended) Dynkin diagram of Ψ we identify the vertices with the roots to which they correspond. For each $\alpha \in \Psi$ we pick a generator e_α for the corresponding root subspace \mathfrak{g}_α and an isomorphism $u_\alpha : k \rightarrow U_\alpha$ to the corresponding root subgroup. For a T -regular closed subgroup H of G we write $\Psi(H) \subseteq \Psi$ for the roots of H relative to T .

Fix a Borel subgroup B of G containing T . Then $\Psi(B)$ is a system of positive roots in Ψ . Let $\Sigma = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots defined by $\Psi(B)$.

Let P be a parabolic subgroup of G . We write P_u for the unipotent radical of P and $\mathfrak{p}_u = \text{Lie}(P_u)$ for the Lie algebra of P_u . Let \mathfrak{n} be a P -submodule of \mathfrak{p}_u . Then P acts on \mathfrak{n} via the adjoint action and on the dual space \mathfrak{n}^* of \mathfrak{n} via the coadjoint action.

As a general reference for the theory of algebraic groups, we cite Borel's book [2]. Throughout we use the labelling of the simple roots for simple G from [3, Planches I–IX].

3. Preliminaries

We discuss the coadjoint action of B on \mathfrak{b}_u^* . We only consider the case when G is simple, the reductive case is similar. There is an isomorphism of G -modules $\phi : \mathfrak{g} \cong \mathfrak{g}^*$ (as \mathfrak{g} and \mathfrak{g}^* have the same highest weight, namely the highest root of Ψ). We fix such an isomorphism ϕ . Restriction from G to B gives \mathfrak{g} and \mathfrak{g}^* the structure of B -modules. We have $\mathfrak{b}_u^* \cong \mathfrak{g}^* / \text{Ann}_{\mathfrak{g}^*}(\mathfrak{b}_u)$ as B -modules, this gives an isomorphism of B -modules $\mathfrak{b}_u^* \cong \mathfrak{g}/\mathfrak{b}$ via ϕ . In the sequel we identify \mathfrak{b}_u^* with $\mathfrak{g}/\mathfrak{b}$ via the above isomorphism.

Next we prove a lemma which we require in Section 4. First we recall the following equation which is a consequence of [2, Theorem AG.10.1]:

$$\dim R = \dim R_x + \dim R \cdot x, \quad (3.1)$$

where R is an algebraic group acting on an algebraic variety X and $x \in X$.

Lemma 3.1. *Suppose the algebraic group $R = HN$ is the semi-direct product of the closed subgroup H and the closed normal subgroup N . Let V be an R -module and let $x \in V$. Then*

$$\dim R \cdot x = \dim H \cdot x + \dim N \cdot x - \dim(H \cdot x \cap N \cdot x).$$

In particular, $H \cdot x \cap N \cdot x$ is finite if and only if $\dim R \cdot x = \dim H \cdot x + \dim N \cdot x$.

Proof. Consider the morphism of algebraic varieties $\psi : H \times (N \cdot x) \rightarrow R \cdot x$, given by $(h, n \cdot x) \mapsto hn \cdot x$. By [2, Theorem AG.10.1], we have

$$\dim R \cdot x = \dim(H \times (N \cdot x)) - \dim \psi^{-1}(x).$$

Elements of $\psi^{-1}(x)$ are of the form $(hk^{-1}, k \cdot x)$ where $h \in H_x$ and $k \in H$ is such that $k \cdot x \in N \cdot x$. Consider the projection of $H \times (N \cdot x)$ onto $N \cdot x$. The image of its restriction to $\psi^{-1}(x)$ is $H \cdot x \cap N \cdot x$. Therefore, $\dim \psi^{-1}(x) = \dim H_x + \dim(H \cdot x \cap N \cdot x)$ (using [2, Theorem AG.10.1]). Then Eq. (3.1) implies that $\dim R \cdot x = \dim H \cdot x + \dim N \cdot x - \dim(H \cdot x \cap N \cdot x)$.

Since $\dim(H \cdot x \cap N \cdot x) = 0$ if and only if $H \cdot x \cap N \cdot x$ is finite, the last part of the lemma follows. \square

We also require the following easy lemma.

Lemma 3.2. *Let R be an algebraic group and let S be a maximal torus of R . Let V be an R -module and let $\lambda_1, \dots, \lambda_j$ be linearly independent weights of V with respect to S . Let v_1, \dots, v_j be eigenvectors of S with weights $\lambda_1, \dots, \lambda_j$, respectively, and let $x = v_1 + \dots + v_j$. Then $S \cdot x = \{t_1 v_1 + \dots + t_j v_j : t_1, \dots, t_j \in k^\times\}$. In particular, $\dim S \cdot x = j$.*

Proof. Since $\lambda_1, \dots, \lambda_j$ are linearly independent, we can find a cocharacter $s_i : k^\times \rightarrow S$ for each i such that

$$s_i(t) \cdot v_{i'} = \begin{cases} t v_i & \text{if } i' = i, \\ v_{i'} & \text{otherwise.} \end{cases}$$

The result follows from considering $s_1(t_1) \dots s_j(t_j) \cdot x$. \square

The next result is about irreducible root systems and is a consequence of the fact that, for N as in the statement, $\Psi(N)$ consists of the positive roots which are not orthogonal to ρ .

Lemma 3.3. *Assume G is a simple algebraic group and let N be the normal subgroup of B generated by the root subgroups of G corresponding to the simple roots which are connected to $-\rho$ in the extended Dynkin diagram of G . Then the map $\alpha \mapsto \rho - \alpha$ is a bijection of $\Psi(N) - \{\rho\}$.*

Proof. Let β_1, \dots, β_r be the simple roots of G , ordered so that β_1, \dots, β_l are connected to $-\rho$ in the extended Dynkin diagram of G (so l is 1 or 2). Let $\alpha = b_1\beta_1 + \dots + b_r\beta_r \in \Psi(B)$. Now $\langle \beta_i, \rho \rangle$ is nonzero if and only if $1 \leq i \leq l$. Therefore, $\langle \alpha, \rho \rangle \neq 0$ if and only if $b_1 + \dots + b_l \neq 0$, i.e. $\alpha \in \Psi(N)$. Therefore, if $\alpha \in \Psi(N) \setminus \{\rho\}$ then $\langle \alpha, \rho \rangle \neq 0$, which in turn implies that $\rho - \alpha \in \Psi(N)$. Also $\rho - \alpha \in \Psi(N)$ means that $\langle \alpha, \rho \rangle \neq 0$ and so $\rho - \alpha \in \Psi(N) \setminus \{\rho\}$. The lemma follows. \square

4. Proofs of Theorem 1.1 and Corollary 1.2

We are now ready to prove Theorem 1.1, the majority of work is done in the following lemma which we prove by induction. In the statement and proof of Lemma 4.1, we abuse notation by identifying root vectors e_α with their image $e_\alpha + \mathfrak{b}$ in \mathfrak{b}_u^* and similarly for linear combinations of root vectors.

Lemma 4.1. *There exists $j \in \mathbb{N}$ and $x \in \mathfrak{b}_u^*$ which satisfy the following conditions.*

- (i) x is of the form $e_{-\beta_1} + \dots + e_{-\beta_j}$ where $\beta_1, \dots, \beta_j \in \Psi(B)$ are linearly independent.
- (ii) $\dim B_u \cdot x \geq \dim \mathfrak{b}_u^* - j$.
- (iii) $B_u \cdot x \cap T \cdot x$ is finite.

Proof. We prove the result by induction on rank G , the base case where G is of type A_1 is trivial. We proceed by induction first noting that for the inductive step we may assume G is simple.

We consider the extended Dynkin diagram of G . Let $I \subseteq \Sigma$ be the set of the simple roots not connected to $-\rho$ in the extended Dynkin diagram. Let H be the closed subgroup of B generated by the maximal torus T and the root subgroups corresponding to the simple roots in I . We note that $H = TC_u$ is a semi-direct product where C is a Borel subgroup of the derived subgroup of a Levi subgroup of the standard parabolic subgroup of G corresponding to I . Inductively we may find $j' \in \mathbb{N}$ and $x' \in \mathfrak{c}_u^*$ of the form $e_{-\beta_1} + \dots + e_{-\beta_{j'}}$ where the β_i are linearly independent,

$$\dim C_u \cdot x' \geq \dim \mathfrak{c}_u - j' \quad (4.1)$$

and $C_u \cdot x' \cap S \cdot x'$ is finite, where S is a maximal torus of C . We define

$$x = x' + e_{-\rho} \in \mathfrak{b}_u^*.$$

Let N be the normal subgroup of B generated by the root subgroups of G corresponding to the simple roots which are connected to $-\rho$ in the extended Dynkin diagram of G (as in Lemma 3.3). We note that $\Psi(B)$ is the disjoint union of $\Psi(N)$ and $\Psi(C)$ and $B_u = C_u N$ is a semi-direct product. Therefore, if $h \in H$, then $h \cdot x$ is of the form

$$h \cdot x = a_\rho e_{-\rho} + \sum_{\alpha \in \Psi(B) \setminus \Psi(N)} a_\alpha e_{-\alpha}. \quad (4.2)$$

We can write $N = \prod_{\alpha \in \Psi(N)} U_{\alpha}$ and an arbitrary element of N in the form $n = \prod_{\alpha \in \Psi(N)} u_{\alpha}(s_{\alpha})$ where we order the α in a way so that the height of α is non-decreasing from right to left. Suppose $n \cdot x \in H \cdot x$ and write $n \cdot x = \sum_{\alpha \in \Psi(B)} b_{\alpha} e_{-\alpha}$. Then we note that $b_{\rho-\beta} = s_{\beta}$ if β is one of the simple roots connected to $-\rho$ in the extended Dynkin diagram of G . Lemma 3.3 implies that $\rho - \beta \in \Psi(N)$ and therefore $s_{\beta} = 0$ since $n \cdot x \in H \cdot x$, i.e., $n \cdot x$ is of the form (4.2). Now consider $\gamma \in \Psi(N)$ of height 2, since $s_{\beta} = 0$ for the simple roots $\beta \in \Psi(N)$, we see that $b_{\rho-\gamma} = s_{\gamma}$ and so $s_{\gamma} = 0$ (again using Lemma 3.3 and the fact that $n \cdot x \in H \cdot x$ so is of the form (4.2)). Continuing this way we can show that $s_{\beta} = 0$ for all $\beta \in \Psi(N) \setminus \{\rho\}$. Therefore, $n \cdot x = x$ and $N \cdot x \cap H \cdot x = \{x\}$. Hence, by Lemma 3.1, we have

$$\dim B \cdot x = \dim H \cdot x + \dim N \cdot x \quad (4.3)$$

and

$$\dim B_u \cdot x = \dim C_u \cdot x + \dim N \cdot x \quad (4.4)$$

since $B_u = C_u N$. An argument similar to that above gives

$$\dim N \cdot x \geq |\Psi(N)| - 1. \quad (4.5)$$

We see that $C_u \cdot x = \{e_{-\rho} + z : z \in C_u \cdot x'\}$ since C_u centralises $e_{-\rho}$. Also $T \cdot x \subseteq \{ae_{-\rho} + z : a \in k^{\times}, z \in S \cdot x'\}$ using Lemma 3.2. Therefore, since $C_u \cdot x' \cap S \cdot x'$ is finite, we get that $C_u \cdot x \cap T \cdot x$ is finite. Thus, Lemma 3.1 gives

$$\dim H \cdot x = \dim C_u \cdot x + \dim T \cdot x. \quad (4.6)$$

From Eqs. (4.1), (4.4) and (4.5) and the fact that C_u centralises $e_{-\rho}$, we get

$$\dim B_u \cdot x \geq \dim \mathfrak{b}_u^* - j, \quad (4.7)$$

where $j = j' + 1$. Equations (4.3), (4.4) and (4.6) give

$$\begin{aligned} \dim B \cdot x &= \dim H \cdot x + \dim N \cdot x = \dim C_u \cdot x + \dim T \cdot x + \dim N \cdot x \\ &= \dim B_u \cdot x + \dim T \cdot x. \end{aligned} \quad (4.8)$$

Then Lemma 3.1 implies that $B_u \cdot x \cap T \cdot x$ is finite. This completes the induction. \square

We now easily deduce Theorem 1.1.

Proof of Theorem 1.1. Let j and x be as in Lemma 4.1. By Lemma 3.2 we get $\dim T \cdot x = j$. Then using Eqs. (4.7) and (4.8), we deduce that $\dim B \cdot x \geq \dim \mathfrak{b}_u^*$ and therefore $\dim B \cdot x = \dim \mathfrak{b}_u^*$. Hence, $B \cdot x$ is dense in \mathfrak{b}_u^* . \square

Table 1
Representatives of dense B -orbits on \mathfrak{b}_u^*

Type	Representative description	
A_r	$\sum_{i=j}^{r-j+1} \alpha_i$	$j = 1, \dots, \lfloor \frac{r}{2} \rfloor$
B_r	$\alpha_j + \sum_{i=j+1}^r 2\alpha_i, \alpha_j$	$j = 1, 3, \dots, r-1$ if r is even
	$\alpha_j + \sum_{i=j+1}^r 2\alpha_i, \alpha_j, \alpha_r$	$j = 1, 3, \dots, r-2$ if r is odd
C_r	$\sum_{i=j}^{r-1} 2\alpha_i + \alpha_r$	$j = 1, \dots, r$
D_r	$\alpha_j + \sum_{i=j+1}^{r-2} 2\alpha_i + \alpha_{r-1} + \alpha_r, \alpha_j, \alpha_{r-1}, \alpha_r$	$j = 1, 3, \dots, r-3$ if r is even
	$\alpha_j + \sum_{i=j+1}^{r-2} 2\alpha_i + \alpha_{r-1} + \alpha_r, \alpha_j, \sum_{i=r-2}^r \alpha_i, \alpha_{r-2}$	$j = 1, 3, \dots, r-4$ if r is odd
G_2	10, 32	
F_4	0100, 0120, 0122, 2342	
E_6	00100, 01110, 11111, 12321	
E_7	000000, 010000, 000100, 000001,	
	012100, 012221, 234321	
E_8	0000000, 0100000, 0001000, 0000010,	
	0121000, 0122210, 2343210, 2465432	

The proofs of Lemma 4.1 and Theorem 1.1 give a representative of a dense B -orbit on \mathfrak{b}_u^* of the form $e_{-\beta_1} + \dots + e_{-\beta_j}$. In Table 1 we describe this representative by listing the roots β_1, \dots, β_j .

Remark 4.2. Let $x \in \mathfrak{b}_u^*$ be as described in Table 1. We have calculated the dimension of the centraliser \mathfrak{b}_x of x in \mathfrak{b} . If $\text{char } k \neq 2$ we have $\dim \mathfrak{b}_x = \dim \mathfrak{b} - \dim \mathfrak{b}_u^*$. It then follows from [2, Proposition 6.7] that the orbit map $B \rightarrow B \cdot x$ is separable. While in the case $\text{char } k = 2$ we have $\dim \mathfrak{b}_x > \dim \mathfrak{b} - \dim \mathfrak{b}_u^*$ (for all simple G) which implies that the orbit map is inseparable.

We now deduce Corollary 1.2. This requires the following two easy lemmas.

Lemma 4.3. Let P be a parabolic subgroup of G and let $\mathfrak{m} \subseteq \mathfrak{n}$ be P -submodules of \mathfrak{p}_u . If \mathfrak{n}^* is a prehomogeneous space for P , then so is \mathfrak{m}^* .

Proof. We have an isomorphism of P -modules $\mathfrak{m}^* \cong \mathfrak{n}^* / \text{Ann}_{\mathfrak{n}^*}(\mathfrak{m})$. Therefore, a dense P -orbit in \mathfrak{n}^* induces a dense P -orbit in \mathfrak{m}^* . \square

Lemma 4.4. Let $P \subseteq Q$ be parabolic subgroups of G and \mathfrak{n} a Q -submodule of \mathfrak{q}_u . If \mathfrak{n}^* is a prehomogeneous space for P , then \mathfrak{n}^* is a prehomogeneous space for Q .

Proof. If $\overline{P \cdot x} = \mathfrak{n}^*$, then $Q \cdot x$ is dense in \mathfrak{n}^* . \square

Proof of Corollary 1.2. By Theorem 1.1 B acts on \mathfrak{b}_u^* with a dense orbit. Therefore, Lemma 4.3 implies \mathfrak{n}^* is a prehomogeneous space for B . Then by Lemma 4.4, P acts on \mathfrak{n}^* with a dense orbit. \square

Remark 4.5. Let $x \in \mathfrak{b}_u^*$ be a representative for a dense B -orbit on \mathfrak{b}_u^* . The proofs of Lemmas 4.3, 4.4 and Corollary 1.2 show that we may take the image of x in \mathfrak{n}^* as a representative of a dense P -orbit on \mathfrak{n}^* and in fact $B \cdot x$ is dense in \mathfrak{n}^* . Therefore, the information given in Table 1 describes a representative of a dense P -orbit on \mathfrak{n}^* . Moreover, it follows from Remark 4.2 that the orbit map $P \rightarrow P \cdot x$ is separable precisely when $\text{char } k \neq 2$.

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